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CLOSED WEINGARTEN HYPERSURFACES IN RIEMANNIAN MANIFOLDS

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0. Introduction

In a complete $(n+1)$-dimensional manifold N we want to find closed hypersurfaces M of *prescribed curvature*, so-called *Weingarten* hypersurfaces. To be more precise, let Ω be a connected open subset of N, $f \in C^{2,\alpha}(\bar{\Omega})$, F a smooth, symmetric function defined in the positive cone $\Gamma_+ \subset \mathbf{R}^n$. Then we look for a convex hypersurface $M \subset \Omega$ such that

$$F|_M = f(x) \quad \forall x \in M, \tag{0.1}$$

where $F|_M$ means that F is evaluated at the vector $(\kappa_i(x))$ the components of which are the principal curvatures of M.

This is in general a problem for a fully nonlinear partial differential equation, which is elliptic if we assume F to satisfy

$$\frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in} \quad \Gamma_+. \tag{0.2}$$

Classical examples of curvature functions F are the elementary symmetric polynomials H_k of order k defined by

$$H_k = \sum_{i_1 < \ldots < i_k} \kappa_{i_1} \ldots \kappa_{i_k}, \quad 1 \le k \le n. \tag{0.3}$$

H_1 is the mean curvature H, H_2 is the scalar curvature - for hypersurfaces in Euclidean space -, and H_n is the Gaussian curvature K.

For technical reasons it is convenient to consider, instead of H_k, the homogeneous polynomials of degree 1

$$\sigma_k = H_k^{1/k}, \tag{0.4}$$

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which are not only monotone increasing but also *concave*. Their *inverses* $\tilde{\sigma_k}$, defined by

$$\tilde{\sigma_k}(\kappa_i) = \frac{1}{\sigma_k(\kappa_i^{-1})}, \tag{0.5}$$

share these properties; a proof of this non-trivial result can be found in [11]. $\tilde{\sigma_1}$ is the so-called *harmonic curvature* G, and, evidently, we have $\tilde{\sigma_n} = \sigma_n$.

The general curvature functions which we have in mind will be defined in Section 1. We shall call those functions to be of class (K); special functions belonging to that class are the n-th root of the Gaussian curvature, the harmonic curvature, the inverse of the *length of the second fundamental form*, i.e.,

$$F(\kappa_i) = \frac{1}{\left(\sum_{i=1}^{n} \kappa_i^{-2}\right)^{1/2}}, \tag{0.6}$$

and, more generally, the inverses of the *complete* symmetric functions $\gamma_k, 1 \le k \le n$, which are homogeneous of degree 1 and defined by

$$\gamma_k(\kappa) = \left(\sum_{|\alpha|=k} \kappa^\alpha\right)^{1/k}. \tag{0.7}$$

Our main assumption in the existence proof is a barrier assumption.

Definition 0.1. Let M_1, M_2 be strictly convex, closed hypersurfaces in N, homeomorphic to S^n and of class $C^{4,\alpha}$ which bound a connected open subset Ω, such that the mean curvature vector of M_1 points outside of Ω and the mean curvature vector of M_2 points inside of Ω. M_1, M_2 are barriers for (F, f) if

$$F|_{M_1} \le f \tag{0.8}$$

and

$$F|_{M_2} \ge f. \tag{0.9}$$

Remark 0.2. In view of the Harnack inequality we deduce from the properties of the barriers that they do not touch, unless both coincide and are solutions of our problem. In this case Ω would be empty.

Then we can prove

Theorem 0.3. *Let the sectional curvature of N be non-positive, let F be of class (K), $0 < f \in C^{2,\alpha}(\bar{\Omega})$ and assume that M_1, M_2 are barriers* *for (F, f). Then the problem*

$$F|_M = f \tag{0.10}$$

has a strictly convex solution $M \subset \bar{\Omega}$ of class $C^{4,\alpha}$.

In a separate paper we shall consider closed Weingarten hypersurfaces in space forms for a class of curvature functions that includes the $\sigma_k' s$, cf.[8].

The existence of closed Weingarten hypersurfaces in $\mathbf{R}^{n+1}$ has been studied extensively by various authors: the case $F = H$ by Bakelman and Kantor [1], Treibergs and Wei [13], the case $F = K$ by Oliker [12], Delanoë [4], and for general curvature functions by Caffarelli, Nirenberg and Spruck [3]. In all the papers - except [4] - the authors imposed a sign condition for the radial derivative of the right-hand side to prove the existence. This condition is necessary for two reasons, first to derive a priori estimates for the C^1-norm and secondly to apply the inverse function theorem, i.e., the kernel of the linearized operator has to be trivial. Without this condition the kernel is no longer trivial, and the inverse function theorem or Leray-Schauder type arguments fail.

We therefore use the evolution method to approximate stationary solutions. But there is still the difficulty of obtaining the C^1-estimates: either one has to impose some artificial condition on the right-hand side, i.e., the condition depends on the choice of a special coordinate system, or one has to stay in the class of convex hypersurfaces where the C^1-estimates are a trivial consequence of the convexity, but then the preservation of the convexity has to be proved and this can only be achieved for special curvature functions such as the Gaussian curvature, or by assuming f to be *concave*; for details see [8].

This paper is organized as follows: In Section 1 we define the curvature functions of class (K) and give sufficient conditions for a curvature function to belong to that class; cf. Lemma 1.4.

In Section 2 we formulate the evolution problem and prove the short-time existence.

Section 3 contains the derivation of the evolution equation for some geometric quantities such as the metric and the second fundamental form.

In Section 4 we demonstrate that the geometric setting can be lifted isometrically to the universal cover, so that without loss of generality we may assume that N is simply connected.

The flow staying in $\bar{\Omega}$ in Section 5 is proved and a priori estimates in the C^1-norm are derived in Section 6.

In Section 7 we obtain the parabolic equations satisfied by h_{ij} resp. $v = \sqrt{1+|Du|^2}$.

In Section 8 the C^2-estimates are derived, while in Section 9 the convergence to a smooth stationary solution is proved.

1. Curvature functions

Let $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar\Gamma_+)$ be a symmetric function satisfying the condition

$$F_i = \frac{\partial F}{\partial \kappa_i} > 0. \tag{1.1}$$

Then F can also be viewed as a function defined on the space of symmetric, positive definite matrices $\mathcal{S}_+$, or to be more precise, at least in this section, let $(h_{ij}) \in \mathcal{S}_+$ with eigenvalues $\kappa_i, 1 \le i \le n$ then define $\hat F$ on $\mathcal{S}_+$ by

$$\hat F(h_{ij}) = F(\kappa_i). \tag{1.2}$$

It is well known, see e.g. [2], that $\hat F$ is as smooth as F and that $\hat F^{ij} = \frac{\partial F}{\partial h_{ij}}$ satisfies

$$\hat F^{ij}\xi_i\xi_j = \frac{\partial F}{\partial \kappa_i}|\xi_i|^2, \tag{1.3}$$

where we use the summation convention throughout this paper unless otherwise stated.

Moreover, if F is concave or convex, then $\hat F$ is also concave or convex, i.e.,

$$\hat F^{ij,kl}\eta_{ij}\eta_{kl} \le 0 \quad \text{or} \quad \hat F^{ij,kl}\eta_{ij}\eta_{kl} \ge 0 \tag{1.4}$$

for any symmetric (η_{ij}), where

$$\hat F^{ij,kl} = \frac{\partial^2}{\partial h_{ij}\partial h_{kl}}\hat F. \tag{1.5}$$

An even sharper estimate is valid, namely,

Lemma 1.1. *Let $F, \hat F$ be defined as above. Then*

$$\hat F^{ij,kl}\eta_{ij}\eta_{kl} = \frac{\partial^2 F}{\partial\kappa_i\partial\kappa_j}\eta_{ii}\eta_{jj} + \sum_{i\ne j}\frac{F_i - F_j}{\kappa_i - \kappa_j}(\eta_{ij})^2, \tag{1.6}$$

for any $(\eta_{ij}) \in \mathcal{S}$, where $\mathcal{S}$ is the space of all symmetric matrices and $F_i = \frac{\partial F}{\partial \kappa_i}$. The second term on the right-hand side of (1.6) *is non-positive* *if F is concave and non-negative if it is convex, and has to be interpreted as a limit if $\kappa_i = \kappa_j$.*

In [6, Lemma 2] it is shown that

$$\left(\frac{\partial F}{\partial \kappa_i} - \frac{\partial F}{\partial \kappa_j}\right)(\kappa_i - \kappa_j) \leq 0 \tag{1.7}$$

if F is concave and that the reverse inequality holds in case F it is convex. Hence the second term on the right-hand side of (1.6) is non-positive or non-negative.

The proof of (1.6) is very elementary but rather lengthy, so we shall only indicate the main steps.

We also want to mention that F need not be defined on the positive cone, and that any open, convex cone will do.

Proof of Lemma 1.1. First, let us remark that by continuity we may assume the eigenvalues of the matrix (h_{ij}), where F is evaluated, to be simple; if not, we can approximate (h_{ij}) by matrices with this property. Let κ_i be the eigenvalues of (h_{ij}), and ${}^i\xi = \left({}^i\xi_k\right)$ the corresponding eigenvectors. Let ${}^r\xi, {}^s\xi$ be two eigenvectors. Then we define the matrix $[r, s]$ by

$$[r, s]_{ij} = \frac{1}{2}\{{}^r\xi_i\,{}^s\xi_j + {}^r\xi_j\,{}^s\xi_j\}. \tag{1.8}$$

We want to evaluate terms such as

$$\hat{F}^{ij,kl}[r_1, r_2]_{ij}[r_3, r_4]_{kl}. \tag{1.9}$$

For simplicity we restrict the ranges of $r_1, ..., r_4$ to $\{1, ..., 4\}$, i.e., $[1, 1]$ represents a generic pair $[r_1, r_1]$, and $[1, 2]$ a generic pair $[r_1, r_2]$ with $r_1 \neq r_2$.

We shall consider several cases.

1. Case. Let us first consider a perturbation

$$\tilde{h}_{ij} = h_{ij} + \varepsilon[1, 2]_{ij}. \tag{1.10}$$

The new non-trivial eigenvalues are

$$\begin{aligned} \tilde{\kappa}_1 &= \tfrac{\kappa_1+\kappa_2}{2} + \sqrt{\tfrac{(\kappa_1-\kappa_2)^2}{4} + \tfrac{\varepsilon^2}{4}}, \\ \tilde{\kappa}_2 &= \tfrac{\kappa_1+\kappa_2}{2} - \sqrt{\tfrac{(\kappa_1-\kappa_2)^2}{4} + \tfrac{\varepsilon^2}{4}}. \end{aligned} \tag{1.11}$$

Let R be the square root on the right-hand side. Then

$$\hat{F}^{ij}\,{}^1\xi_i\,{}^2\xi_j = \hat{F}^{ij}[r_1, r_2]_{ij} = \frac{\partial F}{\partial \tilde{\kappa}_1}\frac{1}{R}\frac{\varepsilon}{4} - \frac{\partial F}{\partial \tilde{\kappa}_2}\frac{1}{R}\frac{\varepsilon}{4} \tag{1.12}$$

and

$$\hat{F}^{ij,kl}[1,2]_{ij}[1,2]_{kl|_{\varepsilon=0}} = \frac{1}{2}\frac{F_1 - F_2}{\kappa_1 - \kappa_2} = 2\frac{F_1 - F_2}{\kappa_1 - \kappa_2}([1,2]_{12})^2$$

$$= \sum_{i \neq j} \frac{F_i - F_j}{\kappa_i - \kappa_j}([1,2]_{ij})^2. \tag{1.13}$$

2. Case. Choose

$$\tilde{h}_{ij} = h_{ij} + \varepsilon[1,2]_{ij} + \delta[3,4]_{ij}, \tag{1.14}$$

and conclude from (1.12),

$$\hat{F}^{ij,kl}[1,2]_{ij}[3,4]_{kl} = 0. \tag{1.15}$$

3. Case. Choose

$$\tilde{h}_{ij} = h_{ij} + \varepsilon[1,2]_{ij} + \delta[3,3]_{ij}, \tag{1.16}$$

and use the same arguments to obtain

$$\hat{F}^{ij,kl}[1,2]_{ij}[3,3]_{kl} = 0. \tag{1.17}$$

4. Case. Choose

$$\tilde{h}_{ij} = h_{ij} + \varepsilon[1,1]_{ij}, \tag{1.18}$$

and deduce

$$\hat{F}^{ij,kl}[1,1]_{ij}[1,1]_{kl} = \frac{\partial^2 F}{\partial\kappa_1\partial\kappa_1} = \frac{\partial^2 F}{\partial\kappa_j\partial\kappa_j}[1,1]_{ii}[1,1]_{jj}. \tag{1.19}$$

5. Case. Choose

$$\tilde{h}_{ij} = h_{ij} + \varepsilon[1,1]_{ij} + \delta[2,2]_{ij}, \tag{1.20}$$

and deduce

$$\hat{F}^{ij,kl}[1,1]_{ij}[2,2]_{kl} = \frac{\partial^2 F}{\partial\kappa_1\partial\kappa_2} = \frac{\partial^2 F}{\partial\kappa_i\partial\kappa_j}[1,1]_{ii}[2,2]_{jj}. \tag{1.21}$$

6. Case. Choose

$$\tilde{h}_{ij} = h_{ij} + \varepsilon[1,1]_{ij} + \delta[1,2]_{ij}, \tag{1.22}$$

and deduce from (1.12),

$$\hat{F}^{ij,kl}[1,1]_{ij}[1,2]_{kl} = 0. \tag{1.23}$$

7. Case. Choose

$$\tilde{h}_{ij} = h_{ij} + \varepsilon[1,2]_{ij} + \delta[1,3]_{ij}. \tag{1.24}$$

The three non-trivial eigenvalues are the solutions of the cubic equation

$$\frac{\delta^2}{4}(\kappa-\kappa_2)+\frac{\varepsilon^2}{4}(\kappa-\kappa_3)-(\kappa-\kappa_1)(\kappa-\kappa_2)(\kappa-\kappa_3)=0. \tag{1.25}$$

They depend smoothly on the parameters ε, δ, and we deduce from (1.25),

$$\frac{\partial\kappa}{\partial\varepsilon}=\frac{\partial\kappa}{\partial\delta}=\frac{\partial^2\kappa}{\partial\varepsilon\partial\delta}=0 \tag{1.26}$$

at $\varepsilon=\delta=0$, where κ represents any of the three eigenvalues. Hence, we obtain

$$\hat{F}^{ij,kl}[1,2]_{ij}[1,3]_{kl}=0. \tag{1.27}$$

Now, let $(\eta_{ij})\in\mathcal{S}$, then

$$(\eta_{ij})=\sum_{r,s}\eta_{rs}[r,s]\equiv\eta_{rs}[r,s] \tag{1.28}$$

and we conclude from the previous particular results

$$\begin{aligned}\hat{F}^{ij,kl}\eta_{ij}\eta_{kl}&=\hat{F}^{ij,kl}[r,s]_{ij}[p,q]_{kl}\eta_{rs}\eta_{pq}\\&=\sum_{p,r}\hat{F}^{ij,kl}[r,r]_{ij}[p,p]_{kl}\eta_{rr}\eta_{pp}\\&\quad+2\sum_{r\neq s}\hat{F}^{ij,kl}[r,s]_{ij}[r,s]_{kl}(\eta_{rs})^2\\&=\frac{\partial^2F}{\partial\kappa_i\partial\kappa_j}\eta_{ii}\eta_{jj}+2\sum_{i\neq j}\left\{\frac{F_i-F_j}{\kappa_i-\kappa_j}\sum_{r\neq s}(\eta_{rs}[r,s]_{ij})^2\right\}\\&=\frac{\partial^2F}{\partial\kappa_i\partial\kappa_j}\eta_{ii}\eta_{jj}+\sum_{i\neq j}\frac{F_i-F_j}{\kappa_i-\kappa_j}(\eta_{ij})^2.\end{aligned} \tag{29}$$

Definition 1.2. A symmetric function $F\in C^0(\bar{\Gamma}_+)\cap C^{2,\alpha}(\Gamma_+)$ homogeneous of degree 1 is said to be of class (K) if

$$F_i=\frac{\partial F}{\partial\kappa_i}>0 \quad \text{in} \quad \Gamma_+, \tag{1.30}$$

$$F \textit{ is concave}, \tag{1.31}$$

$$F|_{\partial\Gamma_+}=0, \tag{1.32}$$

and

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl}\le 2F^{-1}\left(\hat{F}^{ij}\eta_{ij}\right)^2-\hat{F}^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl}, \quad \forall\eta\in\mathcal{S}, \tag{1.33}$$

where $\hat{F}$ is evaluated at $(h_{ij}) \in \mathcal{S}_+$ and $\left(\tilde{h}^{ij}\right) = (h_{ij})^{-1}$.

We immediately deduce from (1.33),

Lemma 1.3. *Let F be of class (K), and κ_m be the largest eigenvalue of $(h_{ij}) \in \mathcal{S}_+$. Then for any $(\eta_{ij}) \in \mathcal{S}$ we have*

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl} \le 2F^{-1}\left(\hat{F}^{ij}\eta_{ij}\right)^2 - \kappa_m^{-1}\hat{F}^{ij}\eta_{im}\eta_{jm}, \tag{1.34}$$

where $\hat{F}$ is evaluated at (h_{ij}) .

For the rest of the paper we shall no longer distinguish between F and $\hat{F}$, instead we shall consider F to be defined on both $\mathcal{S}_+$ and Γ_+ .

Lemma 1.4. *Let $F \in C^0(\bar{\Gamma}_+) \cap C^{2,\alpha}(\Gamma_+)$ be symmetric, homogeneous of degree 1, monotone increasing and convex. Then, its inverse $\tilde{F}$ is of class (K).*

Proof. We first show that $\tilde{F}$ is concave.

We have $\tilde{F}(\kappa_i) = \frac{1}{F(\kappa_i^{-1})}$ so that

$$\tilde{F}_i = F^{-2}F_i\kappa_i^{-2}, \tag{1.35}$$

$$\tilde{F}_{ij} = 2F^{-3}F_iF_j\kappa_i^{-2}\kappa_j^{-2} - F^{-2}F_{ij}\kappa_i^{-2}\kappa_j^{-2} - 2F^{-2}F_i\kappa_i^{-3}\delta_{ij}, \tag{1.36}$$

and therefore, we obtain

$$\tilde{F}_{ij}\xi^i\xi^j \le 2F^{-3}\left(F_i\kappa_i^{-2}\xi^i\right)^2 - 2F^{-2}F_i\kappa_i^{-3}|\xi^i|^2. \tag{1.37}$$

We further estimate

$$\begin{aligned} F_i\kappa_i^{-2}\xi^i &= F_i^{1/2}\kappa_i^{-1/2}F_i^{1/2}\kappa_i^{-3/2}\xi^i \\ &\le \left(F_i\kappa_i^{-1}\right)^{1/2}\left(F_i\kappa_i^{-3}|\xi^i|^2\right)^{1/2}, \end{aligned} \tag{1.38}$$

and conclude that the right-hand side of (1.37) is non-positive, where we have used in addition the homogeneity of F.

Next, we prove that $\tilde{F}$ satisfies the condition (1.33). Let $(h_{ij}) \in \mathcal{S}_+$, $\left(\tilde{h}^{ij}\right) = (h_{ij})^{-1}$ and

$$\tilde{F}(h_{ij}) = \frac{1}{F(\tilde{h}^{ij})}. \tag{1.39}$$

Then,

$$\tilde{F}^{ij} = F^{-2}F_{rs}\tilde{h}^{ra}\tilde{h}^{bs}\frac{\partial h_{ab}}{\partial h_{ij}} = \tilde{F}^2F_{rs}\frac{1}{2}\{\tilde{h}^{ri}\tilde{h}^{js} + \tilde{h}^{rj}\tilde{h}^{is}\}, \tag{1.40}$$

$$\begin{aligned}
\tilde F^{ij,kl} =&2\tilde F^{-1}\tilde F^{ij}\tilde F^{kl}\\
&-\tilde F^2 F_{rs,pq}\frac14\{\tilde h^{ri}\tilde h^{js}+\tilde h^{rj}\tilde h^{is}\}\{\tilde h^{pk}\tilde h^{lq}+\tilde h^{pl}\tilde h^{kq}\}\\
&-\tilde F^2 F_{rs}\frac14\Big[\{\tilde h^{rk}\tilde h^{li}+\tilde h^{rl}\tilde h^{ki}\}\tilde h^{js}+\{\tilde h^{jk}\tilde h^{ls}+\tilde h^{jl}\tilde h^{ks}\}\tilde h^{ri}\\
&\qquad+\{\tilde h^{rk}\tilde h^{lj}+\tilde h^{rl}\tilde h^{kj}\}\tilde h^{is}\\
&\qquad+\{\tilde h^{ik}\tilde h^{ls}+\tilde h^{il}\tilde h^{ks}\}\tilde h^{rj}\Big].
\end{aligned} \tag{1.41}$$

The last term in (1.41) is equal to

$$-\frac12\{\tilde F^{jk}\tilde h^{il}+\tilde F^{ik}\tilde h^{jl}+\tilde F^{jl}\tilde h^{ik}+\tilde F^{il}\tilde h^{jk}\}, \tag{1.42}$$

and thus we deduce

$$\tilde F^{ij,kl}\eta_{ij}\eta_{kl}\le 2\tilde F^{-1}\left(\tilde F^{ij}\eta_{ij}\right)^2-2\tilde F^{ik}\tilde h^{jl}\eta_{ij}\eta_{kl},\quad \forall\eta\in\mathcal S. \tag{1.43}$$

The remaining conditions which functions of class (K) have to satisfy are easily verified.

Remark 1.5.

(i) The mean curvature, the length of the second fundamental form and the γ_k satisfy the assumptions of the lemma, hence their inverses are of class (K).

For the mean curvature and the length of the second fundamental form the required properties are obvious, while the non-trivial result for the γ_k can be found in [11, p.105].

(ii) A straightforward computation shows that the n-th root of the Gaussian curvature is of class (K).

The preceding considerations are also applicable if the κ_i are the principal curvatures of a hypersurface M with metric (g_{ij}). F can then be looked at as being defined on the space of all symmetric tensors (h_{ij}) with eigenvalues κ with respect to the metric. Moreover,

$$F^{ij}=\frac{\partial F}{\partial h_{ij}} \tag{1.44}$$

is a contravariant tensor of second order. Sometimes, it will be convenient to circumvent the dependence on the metric by considering F to depend on the mixed tensor

$$h^i_j=g^{ik}h_{kj}. \tag{1.45}$$

Thus

$$F_i^j = \frac{\partial F}{\partial h_j^i} \tag{1.46}$$

is also a mixed tensor with contravariant index j and covariant index i.

2. The evolution problem

Let N be a complete $(n+1)$-dimensional Riemannian manifold, and M a closed hypersurface. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by $(g_{ij}), (R_{ijkl})$, etc.. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N (resp. M) will be denoted by (x^α) (resp. (ξ^i)). Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e., for a function u on N, (u_α) will be the gradient, and $(u_{\alpha\beta})$ the Hessian, but, e.g. the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\varepsilon}$. We also point out that

$$\bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\varepsilon} x_i^\varepsilon \tag{2.1}$$

with obvious generalizations to other quantities.

In local coordinates x^α and ξ^i the geometric quantities of the hypersurface M are connected by the following equations

$$x_{ij}^\alpha = -h_{ij}\nu^\alpha, \tag{2.2}$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.,

$$x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma, \tag{2.3}$$

where the comma indicates ordinary partial derivatives.

In the implicit definition (2.2) the *second fundamental form* (h_{ij}) is taken with respect to $-\nu$.

The second equation is the *Weingarten equation*

$$\nu_i^\alpha = h_i^k x_k^\alpha, \tag{2.4}$$

where we remember that ν_i^α is a full tensor.

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta \tag{2.5}$$

and the *Gauß equation*

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_\kappa^\gamma x_l^\delta. \tag{2.6}$$

We want to prove that the equation

$$F = f \tag{2.7}$$

has a solution. For technical reasons it is convenient to solve instead of (2.7) the equivalent equation

$$\Phi(F) = \Phi(f), \tag{2.8}$$

where Φ is real function defined on $\mathbf{R}_+$ such that

$$\dot{\Phi} > 0 \quad \text{and} \quad \ddot{\Phi} \leq 0. \tag{2.9}$$

For notational reasons let us abbreviate

$$\tilde{f} = \Phi(f). \tag{2.10}$$

To solve (2.8), we look at the evolution problem

$$\begin{aligned} \dot{x} &= -(\Phi - \tilde{f})\nu, \\ x(0) &= x_0, \end{aligned} \tag{2.11}$$

where x_0 is an embedding of an initial strictly convex hypersurface M_0 diffeomorphic to S^n, $\Phi = \Phi(F)$, and F is evaluated for the principal curvatures of the flow hypersurfaces $M(t)$, or, equivalently, we may assume that F depends on the second fundamental form (h_{ij}) and the metric (g_{ij}) of $M(t)$; $x(t)$ is the embedding for $M(t)$.

This is a parabolic problem, so short-time existence is guaranteed - an exact proof is given below-, and under suitable assumptions we shall be able to prove that the solution exists for all time and that the velocity tends to zero if t goes to infinity.

Consider now a tubular neighbourhood $\mathcal{U}$ of the initial hypersurface M_0. Then we can introduce so-called *normal Gaussian coordinates* x^α, such that the metric in $\mathcal{U}$ has the form

$$d\bar{s}^2 = dr^2 + \bar{g}_{ij}dx^i dx^j, \tag{2.12}$$

where $r = x^0, \bar{g}_{ij} = \bar{g}_{ij}(r,x)$; here we have used slightly ambiguous notation.

A point $p \in \mathcal{U}$ can be represented by its signed distance from M_0 and its base point $x \in M_0$, thus $p = (r,x)$.

Let $M \subset \mathcal{U}$ be a hypersurface which is a graph over M_0, i.e.,

$$M = \{(r,x) : r = u(x), x \in M_0\}. \tag{2.13}$$

The induced metric g_{ij} of M can then be expressed as

$$g_{ij} = \bar{g}_{ij} + u_i u_j \tag{2.14}$$

with inverse

$$g^{ij} = \bar{g}^{ij} - \frac{u^i}{v}\frac{u^j}{v}, \tag{2.15}$$

where $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$ and

$$\begin{aligned} u^i &= \bar{g}^{ij}u_j, \\ v^2 &= 1 + \bar{g}^{ij}u_i u_j. \end{aligned} \tag{2.16}$$

The normal vector ν of M then takes the form

$$(\nu^\alpha) = v^{-1}(1, -u^i), \tag{2.17}$$

if x^0 is chosen appropriately.

From the Gauß formula we immediately deduce that the second fundamental form of M is given by

$$v^{-1}h_{ij} = -u_{ij} + \bar{h}_{ij}, \tag{2.18}$$

where

$$\bar{h}_{ij} = \frac{1}{2}\dot{\bar{g}}_{ij} = \frac{1}{2}\frac{\partial \bar{g}_{ij}}{\partial r} \tag{2.19}$$

is the second fundamental form of the level surfaces $\{r = \text{const}\}$, and the second covariant derivatives of u are defined with respect to the induced metric.

At least for small t the hypersurfaces $M(t)$ are graphs over M_0 and the embedding vector looks like

$$\begin{aligned} x^0(t) &= u(t, x^i(t)), \\ x^i(t) &= x^i(t, \xi^i), \end{aligned} \tag{2.20}$$

where the ξ^i are local coordinates for $M(t)$ independent of t.

Furthermore,

$$\dot{x}^0 = \dot{u} = \frac{\partial u}{\partial t} + \dot{x}^i u_i, \tag{2.21}$$

and from (2.11) we conclude

$$\begin{aligned} \dot{x}^0 &= -(\Phi - \tilde{f})v^{-1}, \\ \dot{x}^i &= v^{-1}u^i(\Phi - \tilde{f}). \end{aligned} \tag{2.22}$$

Hence

$$\frac{\partial u}{\partial t} = -(\Phi - \tilde{f})v. \tag{2.23}$$

This is a scalar equation, which can be solved on a cylinder $[0, \varepsilon] \times M_0$ for small ε, if the principal curvatures of the initial hypersurface M_0 are strictly positive. The equation (2.22) for the embedding vector is then a classical ordinary differential equation of the form

$$\dot{x} = \varphi(t, x). \tag{2.24}$$

We have therefore proved

Theorem 2.1. *The evolution problem* (2.11) *has a solution on a small time interval* $[0, \varepsilon]$.

3. The evolution equations of some geometric quantities

In this section we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces $M(t)$ evolve. All time derivatives are *total* derivatives.

Lemma 3.1 (Evolution of the metric).
The metric g_{ij} *of* $M(t)$ *satisfies the evolution equation*

$$\dot{g}_{ij} = -2(\Phi - \tilde{f})h_{ij}. \tag{3.1}$$

Proof. Let ξ^i be local coordinates for $M(t)$. Then

$$g_{ij} = \bar{g}_{\alpha\beta} x_i^\alpha x_j^\beta \tag{3.2}$$

and thus

$$\dot{g}_{ij} = 2\bar{g}_{\alpha\beta} \dot{x}_i^\alpha x_j^\beta. \tag{3.3}$$

On the other hand, differentiating

$$\dot{x}^\alpha = -(\Phi - \tilde{f})\nu^\alpha \tag{3.4}$$

with respect to ξ^i yields

$$\dot{x}_i^\alpha = -(\Phi - \tilde{f})_i \nu^\alpha - (\Phi - \tilde{f})\nu_i^\alpha, \tag{3.5}$$

and the desired result follows from the Weingarten equation.

Lemma 3.2 (Evolution of the normal).
The normal vector ν *evolves according to*

$$\dot{\nu} = \nabla_M(\Phi - \tilde{f}) = g^{ij}(\Phi - \tilde{f})_i x_j. \tag{3.6}$$

Proof. Since ν is a unit normal vector we have $\dot{\nu} \in T(M)$. Furthermore, differentiating

$$0 = \langle \nu, x_i \rangle \tag{3.7}$$

with respect to t, we deduce

$$\langle \dot{\nu}, x_i \rangle = -\langle \nu, \dot{x}_i \rangle = (\Phi - \tilde{f})_i. \tag{3.8}$$

Lemma 3.3 (Evolution of the second fundamental form).
The second fundamental form evolves according to

$$\dot{h}_i^j = (\Phi - \tilde{f})_i^j + (\Phi - \tilde{f}) h_i^k h_k^j + (\Phi - \tilde{f}) \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj} \tag{3.9}$$

and

$$\dot{h}_{ij} = (\Phi - \tilde{f})_{ij} - (\Phi - \tilde{f}) h_i^k h_{kj} + (\Phi - \tilde{f}) \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \tag{3.10}$$

Proof. We use the *Ricci identities* to interchange the covariant derivatives of ν with respect to t and ξ^i

$$\begin{aligned} \frac{d}{dt}(\nu_i^\alpha) &= (\dot{\nu}^\alpha)_i - \bar{R}^\alpha{}_{\beta\gamma\delta} \nu^\beta x_i^\gamma \dot{x}^\delta \\ &= g^{kl}(\Phi - \tilde{f})_{ki} x_l^\alpha + g^{kl}(\Phi - \tilde{f})_k x_{li}^\alpha - \bar{R}^\alpha{}_{\beta\gamma\delta} \nu^\beta x_i^\gamma \dot{x}^\delta. \end{aligned} \tag{3.11}$$

For the second equality we have used (3.6).

On the other hand, in view of the Weingarten equation we obtain

$$\frac{d}{dt}(\nu_i^\alpha) = \frac{d}{dt}(h_i^k x_k^\alpha) = \dot{h}_i^k x_k^\alpha + h_i^k \dot{x}_k^\alpha. \tag{3.12}$$

Multiplying the resulting equation with $\bar{g}_{\alpha\beta} x_j^\beta$ we conclude

$$\dot{h}_i^k g_{kj} - (\Phi - \tilde{f}) h_i^k h_{kj} = (\Phi - \tilde{f})_{ij} + (\Phi - \tilde{f}) \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \tag{3.13}$$

or equivalently (3.9).

To derive (3.10), we differentiate

$$h_{ij} = h_i^k g_{kj} \tag{3.14}$$

with respect to t and use (3.3).

Lemma 3.4 (Evolution of $(\Phi - \tilde{f})$).
The term $(\Phi - \tilde{f})$ evolves according to the equation

$$\begin{aligned} &(\Phi - \tilde{f})' - \dot{\Phi} F^{ij} (\Phi - \tilde{f})_{ij} \\ =&\dot{\Phi} F^{ij} h_{ik} h_j^k (\Phi - \tilde{f}) + \tilde{f}_\alpha \nu^\alpha (\Phi - \tilde{f}) \\ &+ \dot{\Phi} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta (\Phi - \tilde{f}), \end{aligned} \tag{3.15}$$

where

$$(\Phi - \tilde{f})' = \frac{d}{dt}(\Phi - \tilde{f}) \tag{3.16}$$

and

$$\dot{\Phi} = \frac{d}{dr}\Phi(r). \tag{3.17}$$

Proof. When we differentiate F with respect to t it is advisable to consider F as a function of the mixed tensor h^i_j; then we obtain

$$(\Phi - \tilde{f})' = \dot{\Phi} F^i_j h^j_i - \tilde{f}_\alpha \dot{x}^\alpha. \tag{3.18}$$

(3.15) now follows from (3.9) and (3.4).

4. Lifting of the problem to the universal cover

Let us first recall the definition of a strictly convex hypersurface; strictly convex means that the second fundamental form has a sign.

Then we give

Definition 4.1. Let M be a strictly convex, closed hypersurface homeomorphic to S^n. Then ν is the *outward* unit normal if

$$\langle \Delta_M x, \nu \rangle < 0. \tag{4.1}$$

This definition is consistent with the usual definition of the interior of a convex body bounded by M if the sectional curvature of the ambient space N is non-positive, cf. the considerations below.

In the sequel, we shall always assume that the second fundamental form of a strictly convex hypersurface is *positive* definite, i.e., the normal ν in the Gauß formula (2.2) is the *outward* normal.

In this section we want to show that the open set Ω bounded by the barriers M_1, M_2 is a *distinguished* open set, i.e., it can be isometrically lifted to the universal cover $\tilde{N}$.

By assumption, we have $K_N \le 0$, thus the universal cover is diffeomorphic to $\mathbf{R}^{n+1}$, any geodesic in $\tilde{N}$ is minimizing, and the geodesic spheres around a point are strictly convex with respect to the inner normal, cf. [9, pp. 143-163].

Let $M \subset \tilde{N}$ be a strictly convex, closed hypersurface homeomorphic to S^n. Then $\tilde{N}\backslash M$ has two components Ω_- and Ω_+, one of which is bounded and simply connected. Let Ω_- be the bounded component; we call it the interior of M. Then we can prove

Proposition 4.2. *M is star-shaped with respect to any interior point, i.e., let $x_0 \in \Omega_-$; then any geodesic γ emanating from x_0 intersects M exactly once, and let $\dot{\gamma}$ be the tangent vector at that point. Then*

$$\langle \dot{\gamma}, \nu \rangle > 0, \tag{4.2}$$

where ν is the outward normal according to Definition 4.1.

Proof. First, we shall show that $-\nu$ points into Ω_-.

Fix $x_0 \in \Omega_-$ and introduce geodesic polar coordinates x^α around x_0, so that

$$d\bar{s}^2 = dr^2 + \bar{g}_{ij}dx^i dx^j. \tag{4.3}$$

Let $\bar{x} \in M$ be such that

$$r(\bar{x}) = \sup_M r, \tag{4.4}$$

and let ξ^i be local coordinates for M near $\bar{x}$. Then we have at $\bar{x}$

$$0 = r_i = r_\alpha x_i^\alpha \tag{4.5}$$

and

$$0 \geq r_{ij} = r_{\alpha\beta} x_i^\alpha x_j^\beta + r_\alpha x_{ij}^\alpha. \tag{4.6}$$

Here, $r_\alpha = \nu_\alpha$, and the first term on the right-hand side is the second fundamental form of the geodesic sphere through that point and hence positive definite, i.e., in view of the Gauß formula we have

$$h_{ij} \geq r_{\alpha\beta} x_i^\alpha x_j^\beta > 0, \tag{4.7}$$

which proves that Definition 4.1 is consistent with the geometric notion of interior in this case.

Next, let $\bar{x} \in M$ be such that

$$d(x_0, \bar{x}) = \inf\{d(x_0, x) : x \in M\}, \tag{4.8}$$

and let $\gamma_{\bar{x}}$ be the geodesic connecting x_0 and $\bar{x}$, and $[x_0, \bar{x})$ be its half-open segment. Then

$$[x_0, \bar{x}) \subset \Omega_- \tag{4.9}$$

and

$$\langle \dot{\gamma}_{\bar{x}}, \nu \rangle > 0; \tag{4.10}$$

it is obvious, where the last expression has to be evaluated.

Now, let $x \in M$ be arbitrary and let $\Gamma \subset M$ be any curve connecting $\bar{x}$ and x:

$$\Gamma = \{x(t) : 0 \leq t \leq 1\}, \quad x(0) = \bar{x}. \tag{4.11}$$

Define

$$\Lambda = \{t : \langle \dot{\gamma}_{x(t)}, \nu \rangle > 0 \quad \text{and} \quad [x_0, x(t)) \subset \Omega_-\}. \tag{4.12}$$

Then, $\Lambda \neq \emptyset$, since $0 \in \Lambda$, and we shall show that Λ is both open and closed and hence coincides with the interval $[0, 1]$.

(iii) Λ *is open.* If not, then, in view of the uniqueness of the geodesics, we would deduce the existence of a sequence t_k converging to $t_0 \in \Lambda$ such that there are $x_k \in [x_0, x(t_k)] \cap M$ satisfying

$$x_k \to x(t_0) \quad \text{and} \quad \langle \dot\gamma_{x_k}, \nu \rangle \le 0, \tag{4.13}$$

clearly a contradiction.

(iv) Λ *is closed.* Let $t_k \in \Lambda, t_k \to t_0$ and $t_0 \notin \Lambda$. Then, there are two possibilities: First, suppose

$$[x_0, x(t_0)) \cap \Omega_+ \neq \emptyset, \tag{4.14}$$

which implies that

$$[x_0, x(t_k)) \cap \Omega_+ \neq \emptyset \tag{4.15}$$

for all but a finite number of k's, a contradiction.

Thus, we have

$$[x_0, x(t_0)) \subset \Omega_-, \tag{4.16}$$

but

$$\langle \dot\gamma_{x(t_0)}, \nu \rangle = 0. \tag{4.17}$$

Now, choose Riemannian normal coordinates x^α in $x(t_0)$. Then $\gamma_{x(t_0)}$ is contained in $T_{x(t_0)}M$. In a neighbourhood of $x(t_0)$ we can write M as a graph over $T_{x(t_0)}M$:

$$M = \{x^0 = u(x^i)\}. \tag{4.18}$$

If we choose the coordinates such that at $x(t_0)$

$$\frac{\partial}{\partial x_0} = -\nu, \tag{4.19}$$

then we have at $x(t_0)$

$$h_{ij} = u_{ij}, \tag{4.20}$$

where the derivatives of u are ordinary partial derivatives, i.e., the Euclidean Hessian of u is positive definite in a neighbourhood of $x(t_0)$, or equivalently, M is (locally) strictly convex in $\mathbf{R}^{n+1}$. Thus, Ω_- is (locally) completely contained in the half-space defined by $T_{x_0}M$ contradicting (4.16) and the fact that $\gamma_{x(t_0)}$ is contained in $T_{x_0}M$.

Corollary 4.3. *The interior of a strictly convex hypersurface $M \subset \tilde N$ homeomorphic to S^n is convex.*

Let us consider the domain $\Omega \subset N$ bounded by the barriers M_1, M_2. Each barrier is homeomorphic to $S^n, n \ge 2$, so each M_i has a tubular neighbourhood $\mathcal{U}_i$ which is simply connected, i.e., there is a well defined lift to $\tilde{N}$. More precisely, let

$$\pi : \tilde{N} \to N \tag{4.21}$$

be the covering map. Then each $\pi^{-1}(\mathcal{U}_i)$ consists of several disjoint copies such that the restriction of π to each copy is an isometry on $\mathcal{U}$. Let $\tilde{M}_i, \tilde{M}_i'$ be two generic elements of $\pi^{-1}(M_i)$ and let $\langle\tilde{M}_i\rangle, \langle\tilde{M}_i'\rangle$ be the corresponding open convex bodies. Then we have

Lemma 4.4. *Let $\tilde{M}_i \neq \tilde{M}_i'$. Then*

$$\langle\tilde{M}_i\rangle \cap \langle\tilde{M}_i'\rangle = \emptyset. \tag{4.22}$$

Proof. $\tilde{M}_i'$ is the image of $\tilde{M}_i$ under a deck transformation which is an isometry, hence $\langle\tilde{M}_i'\rangle$ is the image of $\langle\tilde{M}_i\rangle$ under the same deck transformation and so the diameters of the convex bodies are the same.

Thus, if $\tilde{M}_i \neq \tilde{M}_i'$ and

$$\langle\tilde{M}_i\rangle \cap \langle\tilde{M}_i'\rangle \neq \emptyset, \tag{4.23}$$

then $\langle\tilde{M}_i\rangle$ is strictly contained in $\langle\tilde{M}_i'\rangle$ or vice versa, but this is impossible since the diameters are the same.

Corollary 4.5. *For each $\langle\tilde{M}_i\rangle, \pi|_{\overline{\langle\tilde{M}_i\rangle}}$ is an isometry. Let $\langle M_i\rangle$ be the images, then*

$$\Omega = \langle M_2\rangle \backslash \overline{\langle M_1\rangle}. \tag{4.24}$$

Proof. The first claim is evident. To prove (4.24) we only have to show

$$\Omega \subset \langle M_2\rangle. \tag{4.25}$$

Let

$$A = \Omega \cap \langle M_2\rangle. \tag{4.26}$$

(i) *A is non-empty*, since the tubular neighbourhood $\mathcal{U}_2$, previously defined, corresponds to a tubular neighbourhood $\tilde{\mathcal{U}}_2$ of $\tilde{M}_2$ and the notions *interior* and *exterior* relative to M_2 and $\tilde{M}_2$ are the same.

(ii) *A is evidently open.*

(iii) *A is closed* in Ω, for let

$$x_k \in A, \quad x_k \to x \in \Omega; \tag{4.27}$$

then we also know $x \in \overline{\langle M_2\rangle}$ but $x \notin M_2$.

Thus, we have proved that $A = \Omega$ since Ω is connected.

Having laid so much groundwork on this context, let us also consider the case where the ambient space N is a space form with positive curvature, and let us show that the problem can still be lifted to the universal cover; without loss of generality we shall assume that $\tilde{N} = S^{n+1}$. The basic definitions are the same as in the preceding considerations.

First, let us quote a result due to Do Carmo and Warner [5]

Theorem 4.6. *Let $M \subset S^{n+1}$ be a strictly convex hypersurface diffeomorphic to S^n. Then M is contained in an open hemisphere and is the boundary of a convex body.*

Actually, Do Carmo and Warner's result is slightly more general, but that is irrelevant in our context.

Since the shortest geodesic between two points in an open hemisphere is unique, Proposition 4.2 remains valid with the obvious restriction that only geodesics contained in the hemisphere are considered; the other former considerations also apply in this situation and we derive the following theorem.

Theorem 4.7. *Suppose that the universal cover of N either is S^{n+1} or has non-positive sectional curvature . Then the data of our problem Ω, M_1, M_2 and f can be lifted to the universal cover $\tilde{N}$, and Ω is the difference of two convex bodies, one of which is contained in the other.*

In the following we shall therefore assume that N is simply connected.

5. Barriers and a priori estimates in the C^0-norm

By assumption the ambient space N has non-positive curvature, and in the preceding section we have shown that we may assume that N is simply connected. Therefore, we can introduce geodesic polar coordinates $(x^\alpha) = (r, x^i) = (r, x)$ around a point in $\langle M_1 \rangle$ such that

$$d\bar{s}^2 = dr^2 + \bar{g}_{ij} dx^i dx^j, \tag{5.1}$$

and the second fundamental form $\bar{h}_{ij}$ of a geodesic sphere $\{r = \text{const}\}$ is uniformly positive definite in $\bar{\Omega}$.

Let $M(t)$ be a solution of the evolution problem (2.11) in a maximal time interval $I = [0, T^*)$ such that the hypersurfaces are strictly convex. Then, in view of Proposition 4.2 each $M(t)$ can be represented as a graph:

$$M(t) = \{(r, x) : r = u(t, x),\ x \in S_0\}, \tag{5.2}$$

where S_0 is a fixed geodesic sphere. The barriers M_i are also graphs of positive functions u_i. We can then prove

Lemma 5.1. *Choose M_0 either M_1 or M_2 as the initial hypersurface. Then for the embedding vector $x = x(t)$ we have*

$$x(t) \in \bar{\Omega}, \qquad \forall t \in I. \tag{5.3}$$

Proof. We shall only consider the case where $M_0 = M_1$. By Lemma 5.2 below we then obtain

$$\Phi - \tilde{f} \leq 0, \qquad \forall t. \tag{5.4}$$

For all t the flow hypersurfaces are the graphs of functions $u(t)$. Then equations (2.23) and (5.4) yield

$$\frac{\partial u}{\partial t} \geq 0, \tag{5.5}$$

i.e., the flow moves into Ω and

$$\inf_{S_0} u_1 \leq u \qquad \forall t. \tag{5.6}$$

Thus, let us assume that for $t = t_0 > 0$ it is the first time that the flow $M(t)$ touches M_2. Let $\bar{x} = x(t_0) = (u(t_0, \xi_0), \xi_0)$ be that point. In a neighbourhood $B_R = B_R(\xi_0)$ of ξ_0 define

$$\varphi = u_2 - u \geq 0, \qquad u = u(t_0, \cdot). \tag{5.7}$$

Now, because of (5.4) u satisfies the inequality

$$\Phi - \tilde{f} \leq 0 \quad \text{in} \quad B_R, \tag{5.8}$$

and u_2 the reverse inequality

$$\Phi - \tilde{f} \geq 0 \quad \text{in} \quad B_R, \tag{5.9}$$

since M_2 is an upper barrier. Here, we note, that the elliptic operator in the above inequalities is evaluated at u and u_2 respectively.

We then conclude- if we choose B_R small-, that φ satisfies a linearized elliptic inequality of the form

$$-a^{ij}\varphi_{ij} + b^i\varphi_i + c\varphi \geq 0 \quad \text{in} \quad B_R. \tag{5.10}$$

Since φ is nonnegative, the Harnack inequality tells us that φ has to vanish identically in B_R, i.e., if the flow touches M_2 at $t = t_0$, then $M(t_0) = M_2$ and M_2 is a solution of the problem (2.8). The flow is then stationary for $t > t_0$.

Lemma 5.2. *Let $M(t)$ be a solution of the evolution problem* (2.11) *defined on a maximal interval $[0, T^*)$. As the initial hypersurface M_0 we choose either M_1 or M_2; then we obtain*

$$\Phi - \tilde{f} \leq 0 \qquad \forall t \tag{5.11}$$

if $M_0 = M_1$, *and*

$$\Phi - \tilde{f} \geq 0 \qquad \forall t \tag{5.12}$$

if $M_0 = M_2$.

Proof. In Lemma 3.4 we have shown that $\Phi - \tilde{f}$ satisfies a linear parabolic equation; therefore, the proclaimed estimates follow from the maximum principle, since the inequalities are satisfied initially at $t = 0$.

6. A priori estimates in the C^1-norm

The result of Lemma 5.1 implies

$$\inf_{S_0} u_1 \leq u \leq \sup_{S_0} u_2. \tag{6.1}$$

We shall show that Du and hence the induced metric of $M(t)$ is uniformly bounded, cf. (2.14), as long as the $M(t)$ remain convex.

Lemma 6.1. *Let* $M = \operatorname{graph} u|_{S_0}$ *be a closed convex hypersurface represented in normal Gaussian coordinates. Then the quantity* $v = \sqrt{1 + |Du|^2}$ *can be estimated by*

$$u \leq c(|u|, S_0, \bar{g}_{ij}). \tag{6.2}$$

Proof. We have

$$g_{ij} = \bar{g}_{ij} + u_i u_j, \qquad \bar{g}_{ij} = \bar{g}_{ij}(u, x). \tag{6.3}$$

Define

$$\|Du\|^2 = g^{ij} u_i u_j, \qquad |Du|^2 = \bar{g}^{ij} u_i u_j. \tag{6.4}$$

then

$$\|Du\|^2 = \frac{|Du|^2}{v^2} \tag{6.5}$$

and

$$v^{-2} = 1 - \| Du \|^2 . \tag{6.6}$$

Let φ be defined by

$$\varphi = \log v + \lambda u, \tag{6.7}$$

where the parameter λ will be chosen later, and let $x_0 \in S_0$ be such that

$$\varphi(x_0) = \sup_{S_0} \varphi. \tag{6.8}$$

Then, we have at x_0

$$0 = \varphi_i = v^{-1} v_i + \lambda u_i \tag{6.9}$$

or

$$0 = v^{-1} v_i u^i + \lambda \|Du\|^2. \tag{6.10}$$

Differentiating v yields

$$v_i = u_{ij} u^i v^3, \tag{6.11}$$

i.e.,

$$v_i u^i = u_{ij} u^i u^j v^3. \tag{6.12}$$

We then conclude from (2.18),

$$0 = -h_{ij} u^i u^j v^2 + \lambda \|Du\|^2 + \bar{h}_{ij} u^i u^j v^2. \tag{6.13}$$

We now observe that

$$u^i = g^{ij} u_j = \bar{g}^{ij} u_j v^{-2}. \tag{6.14}$$

Let $\bar{\kappa}$ be an upper bound for the eigenvalues of $\bar{h}_{ij}$. Then

$$\bar{h}_{ij} u^i u^j v^2 \le \bar{\kappa} v^{-2} |Du|^2, \tag{6.15}$$

and in view of (6.13) we deduce

$$0 \le (\bar{\kappa} + \lambda) |Du|^2 v^{-2} \tag{6.16}$$

at x_0.

Let us now choose $\lambda = -\bar{\kappa} - \varepsilon$. Then $Du = 0$ and

$$\varphi \le \varphi(x_0) = \lambda u(x_0), \tag{6.17}$$

or equivalently

$$v \le e^{\lambda\{u(x_0) - u\}} \le e^{|\lambda|\{\sup u - \inf u\}}. \tag{6.18}$$

By letting ε tend to zero we finally obtain

$$v \le e^{\bar{\kappa}\{\sup u - \inf u\}}. \tag{6.19}$$

7. The evolution equations for h_{ij} and v

Let us first derive the parabolic equation for the second fundamental form.

Lemma 7.1. *Let $M(t)$ be a solution of the problem (2.11). Then the second fundamental form satisfies*

$$
\begin{aligned}
\dot h_{ij} - \dot\Phi F^{kl} h_{ij;kl} =&\dot\Phi F^{kl} h_{kr} h^r_l h_{ij} - (\Phi - \tilde f) h^k_i h_{kj} - \dot\Phi F h^k_i h_{kj}\\
&- \tilde f_{\alpha\beta} x^\alpha_i x^\beta_j + \tilde f_\alpha \nu^\alpha h_{ij} + \ddot\Phi F_i F_j\\
&+ \dot\Phi F^{kl,rs} h_{kl;i} h_{rs;j}\\
&+ (\Phi - \tilde f) \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_i \nu^\gamma x^\delta_j + 2\dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_i x^\gamma_k x^\delta_j h^r_l\\
&- \dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_k x^\gamma_i x^\delta_l h^r_j - \dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_k x^\gamma_j x^\delta_l h^r_i\\
&+ \dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_k \nu^\gamma x^\delta_l h_{ij} - \dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta} \{\nu^\alpha x^\beta_i \nu^\gamma x^\delta_j\\
&+ \dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta;\epsilon} \{\nu^\alpha x^\beta_k x^\gamma_l x^\delta_i x^\epsilon_j + \nu^\alpha x^\beta_i x^\gamma_k x^\delta_j x^\epsilon_l\}.
\end{aligned}
\tag{7.1}
$$

Proof. We start with equation (3.10) and shall evaluate the term

$$(\Phi - \tilde f)_{ij}. \tag{7.2}$$

First, we have

$$\Phi_i = \dot\Phi F_i = \dot\Phi F^{kl} h_{kl;i} \tag{7.3}$$

and

$$\Phi_{ij} = \dot\Phi F^{kl} h_{kl;ij} + \ddot\Phi F^{kl} h_{kl;i} F^{rs} h_{rs;j} + \dot\Phi F^{kl;rs} h_{kl;i} h_{rs;j}. \tag{7.4}$$

Next, replacing $h_{ij;kl}$ by $h_{ij;kl}$, and differentiating the Codazzi equation

$$h_{kl;i} = h_{ik;l} + \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_k x^\gamma_l x^\delta_i \tag{7.5}$$

yielding

$$
\begin{aligned}
h_{kl;ij} =&h_{ik;lj} + \bar R_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x^\beta_k x^\gamma_l x^\delta_i x^\epsilon_j\\
&+ \bar R_{\alpha\beta\gamma\delta} \{\nu^\alpha_j x^\beta_k x^\gamma_l x^\delta_i + \nu^\alpha x^\beta_{kj} x^\gamma_l x^\delta_i + \nu^\alpha x^\beta_k x^\gamma_{lj} x^\delta_i + \nu^\alpha x^\beta_k x^\gamma_l x^\delta_{ij}\}.
\end{aligned}
\tag{7.6}
$$

To replace $h_{kl;ij}$ by $h_{ij;kl}$ we use the *Ricci* identities

$$h_{ik;lj} = h_{ik;lj} + h_{ak} R^a{}_{ilj} + h_{ai} R^a{}_{klj} \tag{7.7}$$

and differentiate once again the Codazzi equation

$$h_{ik;j} = h_{ij;k} + \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_i x^\gamma_k x^\delta_j. \tag{7.8}$$

To replace $\tilde f_{ij}$ we use the chain rule

$$
\begin{aligned}
\tilde f_i &= \tilde f_\alpha x^\alpha_i,\\
\tilde f_{ij} &= \tilde f_{\alpha\beta} x^\alpha_i x^\beta_j + \tilde f_\alpha x^\alpha_{ij}.
\end{aligned}
\tag{7.9}
$$

Then by the Gauß equation and Gauß formula, the symmetry properties of the Riemann curvature tensor and the homogeneity of F, i.e.,

$$F = F^{kl} h_{kl}, \tag{7.10}$$

we deduce equation (7.1) from (3.10).

Since the mixed tensor h^i_j is a more natural geometric object, let us look at the evolution equation for h^i_i that can be derived from (3.9).

Lemma 7.2. *The evolution equation for h^i_i (no summation over i) has the form*

$$\begin{aligned}\dot h^i_i - \dot\Phi F^{kl} h^i_{i;kl} =& \dot\Phi F^{kl} h_{kr} h^r_l h^i_i + (\Phi - \tilde f) h^k_i h^i_k - \dot\Phi F h^k_i h^i_k \\ &- \tilde f_{\alpha\beta} x^\alpha_i x^\beta_k g^{ki} + \tilde f_\alpha \nu^\alpha h^i_i + \ddot\Phi F_i F^i + \dot\Phi F^{kl,rs} h_{kl;i} h_{rs;m} g^{mi} \\ &+ (\Phi - \tilde f) \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_i \nu^\gamma x^\delta_m g^{mi} \\ &+ 2\dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_i x^\gamma_k x^\delta_m g^{mi} h^r_l - 2\dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_k x^\gamma_i x^\delta_l h^{ri} \\ &+ \dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_k \nu^\gamma x^\delta_l h^i_i - \dot\Phi F \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_i \nu^\gamma x^\delta_m g^{mi} \\ &+ \dot\Phi F^{kl} \bar R_{\alpha\beta\gamma\delta;\varepsilon} \{\nu^\alpha x^\beta_k x^\gamma_l x^\delta_i x^\varepsilon_m + \nu^\alpha x^\beta_i x^\gamma_k x^\delta_m x^\varepsilon_l\} g^{mi}.\end{aligned} \tag{7.11}$$

Let M be a hypersurface that can be written as a graph in a normal Gaussian coordinate system $(x^a) = (r, x^i)$. From the relation (2.17) it follows that

$$v = \sqrt{1 + |Du|^2} = (r_\alpha \nu^\alpha)^{-1}. \tag{7.12}$$

For the hypersurfaces $M(t)$ defined by the flow (2.11) we have

Lemma 7.3. *Consider the flow in a normal Gaussian coordinate system where the $M(t)$ can be written as the graph of a function $u(t)$. Then v safisfies the evolution equation*

$$\begin{aligned}\dot v - \dot\Phi F^{ij} v_{ij} =& - \dot\Phi F^{ij} h_{ik} h^k_j v - 2v^{-1} \dot\Phi F^{ij} v_i v_j \\ &+ r_{\alpha\beta} \nu^\alpha \nu^\beta [(\Phi - \tilde f) - \dot\Phi F] v^2 \\ &+ \dot\Phi F^{ij} \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_i x^\gamma_j x^\delta_k r_\varepsilon x^\varepsilon_m g^{mk} v^2 + 2\dot\Phi F^{ij} r_{\alpha\beta} h^k_i x^\alpha_k x^\beta_j v^2 \\ &+ \dot\Phi F^{ij} r_{\alpha\beta\gamma} \nu^\alpha x^\beta_i x^\gamma_j v^2 + \tilde f_\alpha x^\alpha_m g^{mk} r_\beta x^\beta_k v^2\end{aligned} \tag{7.13}$$

Proof. Differentiating (7.12) gives

$$v_i = -v^2 \{r_{\alpha\beta} \nu^\alpha x^\beta_i + r_\alpha \nu^\alpha_i\}, \tag{7.14}$$

$$\begin{aligned}v_{ij} = 2v^{-1} v_i v_j - v^2 \{& r_{\alpha\beta\gamma} \nu^\alpha x^\beta_i x^\gamma_j + r_{\alpha\beta} \nu^\alpha_j x^\beta_i \\ &+ r_{\alpha\beta} \nu^\alpha x^\beta_{ij} + r_{\alpha\beta} \nu^\alpha_i x^\beta_j + r_\alpha \nu^\alpha_{ij}\}.\end{aligned} \tag{7.15}$$

We also have to calculate the time derivative of v:

$$\begin{aligned} \dot v &= -\{r_{\alpha\beta}\nu^\alpha \dot x^\alpha + r_\alpha \dot\nu^\alpha\} v^2 \\ &= r_{\alpha\beta}\nu^\alpha\nu^\beta(\Phi - \tilde f)v^2 - r_\alpha(\Phi-\tilde f)_k x^\alpha_m g^{mk} v^2, \end{aligned} \tag{7.16}$$

where we have used (3.6).

By substituting (7.15) and (7.16) on the left-hand side of (7.13) and simplifying the resulting expression with the help of the Weingarten and Codazzi equations we arrive at the desired conclusion.

Lemma 7.4. *For convex hypersurfaces which stay in a compact domain we have*

$$|F^{ij} r_{\alpha\beta} h^k_i x^\alpha_k x^\beta_j| \le cF \tag{7.17}$$

Proof. Choose a coordinate system ξ^i such that in a fixed but arbitrary point in M

$$g_{ij} = \delta_{ij}, \qquad h_{ij} = \kappa_i \delta_{ij}. \tag{7.18}$$

Then,

$$\begin{aligned} |F^{ij} r_{\alpha\beta} h^k_i x^\alpha_k x^\beta_j| \le \sum_i |F^{ii} h^i_i| \sup |D^2 r| &= F^{ij} h_{ij} \sup|D^2 r| \\ &= F \sup |D^2 r|. \end{aligned} \tag{7.19}$$

8. A priori estimates in the C^2-norm

Let $M(t)$ be a solution of the evolution problem (2.11) with initial hypersurface $M_0 = M_1$ defined on a maximal time interval $I = [0, T^*)$. We also assume that F is of class (K) as in Definition 1.2, and we choose $\Phi(t) = -t^{-1}$. Let $M(t)$ be represented as the graph of a function u in geodesic polar coordinates. Then, from (2.11) we deduce

$$\dot u = \frac{du}{dt} = -(\Phi - \tilde f)v^{-1}, \tag{8.1}$$

and taking the relation (2.18) into account we conclude

$$\dot u - \dot\Phi F^{ij} u_{ij} = -(\Phi - \tilde f)v^{-1} + \dot\Phi F v^{-1} - \dot\Phi F^{ij} \bar h_{ij}. \tag{8.2}$$

Here, the $\bar h_{ij}$ are uniformly positive definite in $\bar\Omega$, i.e., we can estimate

$$F^{ij}\bar h_{ij} \ge cF^{ij} g_{ij} \ge cF(1, \ldots, 1) \tag{8.3}$$

with a positive constant c. The second estimate in (8.3) follows from

Lemma 8.1. *Let $F \in C^2(\Gamma_+)$ be homogeneous of degree 1, monotone increasing and concave. Then*

$$F^{ij} g_{ij} \geq cF(1, ..., 1). \tag{8.4}$$

A proof can be found in [14, Lemma 3.2].

We first note that in view of Lemma 5.2 we know that

$$\Phi \leq \tilde{f} \quad \text{or} \quad F \leq f, \tag{8.5}$$

and that by the results in Section 5 the flow stays in the compact set $\bar{\Omega}$. Furthermore, due to the choice of Φ and the condition (1.32) the $M(t)$ are strictly convex during the evolution and, hence, Du is uniformly bounded.

An estimate for the second derivatives of u is given in

Lemma 8.2. *Let F be of class (K). Then the principal curvatures of the evolution hypersurfaces $M(t)$ are uniformly bounded.*

Proof. Let φ and w be defined respectively by

$$\varphi = \sup\{h_{ij}\eta^i\eta^j : \|\eta\| = 1\}, \tag{8.6}$$

$$w = \log \varphi + \log v + \lambda u, \tag{8.7}$$

where λ is a large positive parameter. We claim that w is bounded.

Let $0 < T < T^*$, and $x_0 = x(t_0), 0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$\sup_{M_0} w < \sup\{\sup_{M(t)} w : 0 < t \leq T\} = w(x_0). \tag{8.8}$$

We then can introduce a Riemannian normal coordinate system ξ^i at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h^n_n. \tag{8.9}$$

Let $\eta = \left(\eta^i\right)$ be the contravariant vector defined by

$$\eta = (0, ..., 0, 1), \tag{8.10}$$

and set

$$\tilde{\varphi} = \frac{h_{ij}\eta^i\eta^j}{g_{ij}\eta^i\eta^j}. \tag{8.11}$$

$\tilde{\varphi}$ is well defined in a neighbourhood of (t_0, ξ_0).

Now, define $\tilde{w}$ by replacing φ by $\tilde{\varphi}$ in (8.7); then $\tilde{w}$ assumes its maximum at (t_0, ξ_0). Moreover, at (t_0, ξ_0)

$$\dot{\tilde{\varphi}} = \dot{h}^n_n, \tag{8.12}$$

and the spacial derivatives do also coincide; in short, at (t_0, ξ_0) $\tilde{\varphi}$ satisfies the same differential equation (7.11) as h^n_n. For the sake of greater clarity, let us therefore treat h^n_n like a scalar and pretend that w is defined by

$$w = \log h^n_n + \log v + \lambda u. \tag{8.13}$$

At (t_0, ξ_0) we have $\dot{w} \geq 0$, and, in view of the maximum principle, we deduce from (7.11),(7.13) and (8.2)

$$\begin{aligned} 0 \leq & - F^{-1} h^n_n - c(\Phi - \tilde{f}) + c + \dot{\Phi} F^{ij} g_{ij} c - \lambda(\Phi - \tilde{f}) v^{-1} \\ & + \lambda F^{-1} v^{-1} - \lambda \dot{\Phi} F^{ij} \bar{h}_{ij} - \dot{\Phi} F^{ij} (\log v)_i (\log v)_j \\ & + \dot{\Phi} F^{ij} \left(\log h^n_n\right)_i \left(\log h^n_n\right)_j \\ & + \left\{ \ddot{\Phi} F_n F^n + \dot{\Phi} F^{kl,rs} h_{kl,n} h_{rs,m} g^{mn} \right\} \left(h^n_n\right)^{-1}, \end{aligned} \tag{8.14}$$

where we have estimated bounded terms by a positive constant c, assumed that $h^n_n \geq 1$, and also observed (8.5).

Now, the last term in the preceding inequality is estimated from above by

$$- \left(h^n_n\right)^{-2} \dot{\Phi} F^{ij} h_{in;n} h_{jn;m} g^{mn}, \tag{8.15}$$

cf. Lemma 1.3, in view of the choice of Φ. Moreover, because of the Codazzi equation we have

$$h_{in;n} = h_{nn;i} + \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_n x^\gamma_i x^\delta_n, \tag{8.16}$$

and hence, when we abbreviate the curvature term by $\bar{R}_i$, we conclude that (8.15) is equal to

$$- \left(h^n_n\right)^{-2} \dot{\Phi} F^{ij} \left(h^n_{n;i} + \bar{R}_i\right) \left(h^n_{n;j} + \bar{R}_j\right). \tag{8.17}$$

Thus, the terms in (8.14) containing the derivatives are estimated from above by

$$- \dot{\Phi} F^{ij} (\log v)_i (\log v)_j - 2 \left(h^n_n\right)^{-1} \dot{\Phi} F^{ij} \left(\log h^n_n\right)_i \bar{R}_j. \tag{8.18}$$

Moreover, at ξ_0 Dw vanishes, i.e.,

$$D \log h^n_n = - D \log v - \lambda D u, \tag{8.19}$$

and (8.18) is further estimated from above by

$$\left(h^n_n\right)^{-1} c \lambda \dot{\Phi} F^{ij} g_{ij}, \tag{8.20}$$

where we have assumed $\lambda \geq 1$.

Summarizing, we deduce from (8.14)

$$(8.21)\quad \begin{aligned} 0 \leq &\{-F^{-1}h^n_n + c + \lambda F^{-1}v^{-1} - \lambda(\Phi - \tilde{f})v^{-1} - c(\Phi - \tilde{f})\} \\ &+ \{c\dot{\Phi}F^{ij}g_{ij} + (h^n_n)^{-1} c\lambda\dot{\Phi}F^{ij}g_{ij} - \lambda\dot{\Phi}F^{ij}\bar{h}_{ij}\} \end{aligned}$$

We now choose λ very large and assume that

$$(8.22)\qquad h^n_n > \mu,$$

where μ is also large, and we deduce that the terms involving $\dot{\Phi}$ sum up to something negative if we choose μ large. Thus, we conclude that we are left with

$$(8.23)\quad 0 \leq -F^{-1}h^n_n + c + \lambda F^{-1}v^{-1} - \lambda(\Phi - \tilde{f})v^{-1} - c(\Phi - \tilde{f}),$$

i.e., h^n_n and hence w are a priori bounded at (t_0, ξ_0).

To complete the a priori estimates we have to show that the principal curvatures can be bounded from below by a positive constant, or equivalently, since F vanishes on $\partial\Gamma_+$, that F is bounded from below by a positive constant.

Lemma 8.3. *Let F be of class (K). Then there is a positive constant ε_0 such that*

$$(8.24)\qquad \varepsilon_0 \leq F$$

during the evolution.

Proof. Consider the function

$$(8.25)\qquad w = -(\Phi - \tilde{f}) + \lambda u,$$

where λ is large. Let $0 < T < T^*$ and suppose

$$(8.26)\qquad \sup_{M_0} w < \sup\{\sup_{M(t)} w : 0 \leq t \leq T\}.$$

Then, there is $x_0 = x(t_0), 0 < t_0 \leq T$, such that

$$(8.27)\qquad w(x_0) = \sup\{\sup_{M(t)} w : 0 \leq t \leq T\}.$$

From (3.15),(8.2) and the maximum principle we then infer

$$(8.28)\quad \begin{aligned} 0 \leq &-\dot{\Phi}F^{ij}h_{ik}k^k_j(\Phi - \tilde{f}) - \dot{\Phi}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i \nu^\gamma x^\delta_j(\Phi - \tilde{f}) \\ &- \tilde{f}_\alpha\nu_\alpha(\Phi - \tilde{f}) - \lambda(\Phi - \tilde{f})v^{-1} + \lambda\dot{\Phi}Fv^{-1} - \lambda\dot{\Phi}F^{ij}\bar{h}_{ij}. \end{aligned}$$

Let κ be an upper bound for the principle curvatures. Then the first term on the right-hand side of (8.28) can be estimated by

$$(8.29)\quad -\dot{\Phi}F^{ij}h_{ij}\kappa(\Phi - \tilde{f}) = -\dot{\Phi}F\kappa(\Phi - \tilde{f}) = -\kappa F^{-1}(\Phi - \tilde{f});$$

the second term is non-positive because $K_N \le 0$; from the remaining terms the last one is negative and has as dominating factor $\lambda\dot{\Phi}$. Hence F cannot be too small at x_0 and the lemma is proved.

9. Convergence to a stationary solution

We are now ready to prove Theorem 0.3. Let $M(t)$ be the flow with initial hypersurface $M_0 = M_1$. Let us look at the scalar version of the flow (2.23):

$$\frac{\partial u}{\partial t} = -(\Phi - \tilde{f})v. \tag{9.1}$$

This is a scalar parabolic differential equation defined on the cylinder

$$Q_{T^*} = [0, T^*) \times S_0 \tag{9.2}$$

with initial value $u_0 = u_1 \in C^{4,\alpha}(S_0)$. S_0 is a geodesic sphere equipped with the induced metric. In view of the a priori estimates we have proved in the preceding sections, we know that

$$|u|_{2,0,S_0} \le c \tag{9.3}$$

and

$$F \text{ is uniformly elliptic in } u \tag{9.4}$$

independent of t. Furthermore, F is concave and thus we can apply the regularity results in Krylov [10, Chapter 5.5] to conclude that uniform $C^{2,\alpha}$-estimates are valid, leading further to uniform $C^{4,\alpha}$-estimates due to the regularity results for linear operators.

Therefore, the maximal time interval is unbounded, i.e., $T^* = \infty$.

Now, integrating (9.1) and observing that the right-hand side is non-negative we obtain

$$u(t,x) - u(0,x) = -\int_0^t (\Phi - \tilde{f})v \ge -\int_0^t (\Phi - \tilde{f}), \tag{9.5}$$

i.e.,

$$\int_0^\infty |\Phi - \tilde{f}| < \infty \qquad \forall x \in S_0. \tag{9.6}$$

Thus, for any $x \in S_0$ there is a sequence $t_k \to \infty$ such that $(\Phi - \tilde{f}) \to 0$.

On the other hand, $u(\cdot, x)$ is monotone increasing and therefore

$$\lim_{t\to\infty} u(t,x) = \tilde{u}(x) \tag{9.7}$$

exists and is of class $C^{4,\alpha}(S_0)$ in view of the a priori estimates. We finally deduce that $\tilde{u}$ is a stationary solution of our problem and that

$$\lim_{t\to\infty} (\Phi - \tilde{f}) = 0. \tag{9.8}$$

References

[1] I. Bakelman & B. Kantor, *Existence of spherically homeomorphic hypersurfaces in Euclidean space with prescribed mean curvature*, Geometry and Topology, Leningrad, **1** (1974) 3-10.

[2] L. Caffarelli, L. Nirenberg & J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Math. **155** (1985) 261-301.

[3] ______, *Nonlinear second order elliptic equations IV: Starshaped compact Weingarten hypersurfaces*, Current topics in partial differential equations, Y. Ohya, K. Kasahara & N. Shimakura (eds.), Kinokunize, Tokyo, 1985.

[4] P. Delanoë, *Plongements radiaux $S^n \to \mathbf{R}^{n+1}$ á courbure de Gauss positive prescrite*, Ann. Sci. École Norm. Sup. (4) **18** (1985), 635-649.

[5] M. P. Do Carmo & F. W. Warner, *Rigidity and convexity of hypersurfaces in spheres*, J. Differential Geom. **4** (1970) 133-144.

[6] K. Ecker & G. Huisken, *Immersed hypersurfaces with constant Weingarten curvature*, Math. Ann. **283** (1989) 329-332.

[7] C. Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990) 299-314.

[8] ______, *Closed Weingarten hypersurfaces in space forms*, Preprint.

[9] D. Gromoll, W. Klingenberg & W. Meyer, *Riemannsche Geometrie im Großen*, Lecture Notes in Math., Vol. 55, Springer, Berlin, 1975.

[10] N. V. Krylov, *Nonlinear elliptic and parabolic equations of second order*, Reidel, Dordrecht, 1987.

[11] D. S. Mitrinovic, *Analytic inequalities*, Springer, Berlin, 1970.

[12] V. I. Oliker, *Hypersurfaces in $\mathbf{R}^{n+1}$ with prescribed Gaussian curvature and related equations of Monge-Ampére type*, Comm. Partial Differential Equations **9** (1984) 807-838.

[13] A. E. Treibergs & S. W. Wei, *Embedded hypersurfaces with prescribed mean curvature*, J. Differential Geom. **18** (1983) 513-521.

[14] J. I. E. Urbas, *On the expansion of convex hypersurfaces by symmetric functions of their principal radii of curvature*, J. Differential Geom. **33** (1991) 91-125.

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